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On the short surface waves due to an oscillating, partially immersed body

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A body whose boundary C is vertical at the free surface F oscillates at high frequency in some prescribed manner. Asymptotic forms in terms of limit potentials are obtained for the upward force and the moment about a line in F of the pressure forces on the body. The cylindrical waves radiated to infinity are found to be asymptotically equivalent to those obtained in a two-dimensional solution of the Helmholtz equation. Some illustrative examples are given, principally the horizontal ellipsoid, in which case comparison with strip theory is possible.

1. Introduction

Cartesian co-ordinates $(x, y. z)$ are chosen with z measured vertically downwards and origin in the plane of the undisturbed free surface F of infinitely deep, inviscid, incompressible fluid under gravity. The boundary C of a partially immersed body containing the origin oscillates in a prescribed manner with small constant amplitude and period $2\pi/\sigma$ about the equilibrium position. Surface tension is neglected and the motion is assumed small enough for the equations to be linearized. Then the velocity potential, which is of the form $\mathrm{Re}\{\phi(x, y, z)\, e^{-i\sigma t}\}$, satisfies

$$\nabla^2\phi = 0 \tag{1.1}$$

throughout the fluid and the boundary conditions

$$K\phi + \partial\phi/\partial z = 0 \quad \text{on} \quad F, \tag{1.2}$$

where $K = \sigma^2/g$ and g is the gravitational acceleration, and

$$\partial\phi/\partial n = u, \text{ a given complex velocity distribution, on } C, \tag{1.3}$$

where $\partial/\partial n$ denotes the normal derivative directed into the fluid. Furthermore, ϕ must satisfy a radiation condition, namely that only outgoing waves exist at infinity. John (1950) showed, for finite or infinite depth, that if the body is entirely contained within the vertical cylinder whose cross-section in the plane of F coincides with that of C, then the above boundary-value problem for ϕ possesses a unique solution. On physical considerations, it seems likely that this restriction on the shape of the body is not crucial in this respect and that the solution is uniquely determined not only for any smooth body but also when the body is not simply connected, e.g. a torus, or when more than one body is generating the wave motion.

Another variation of the problem occurs when the fluid is no longer of infinite extent but is contained in a channel or lake. Then no energy can be radiated to infinity but normal modes of oscillation exist and resonance may occur. Examples of this and the multi-body problem will be discussed in a later paper.

Here it is of interest to find the amplitude and phase of the waves radiated to infinity. This will be done in terms of the velocity potential; the surface elevation Re $[\xi(x,y)\,e^{-i\sigma t}]$ is then obtained from Bernoulli's equation, used to establish (1.2), and is given by

$$\xi(x,y) = (iK/\sigma)\,\phi(x,y,0). \tag{1.4}$$

The most commonly considered prescribed motion is that of heaving, in which case the function u in (1.3) is $U\,\partial z/\partial n$, where U is the amplitude of the heave velocity, and another quantity of interest is the virtual or added mass M, given by

$$M = -\frac{\rho}{U^2}\,\mathrm{Re}\int_C \phi\,\frac{\partial\phi}{\partial n}\,dS, \tag{1.5}$$

where ρ is the density of the fluid. The imaginary part of the same integral expression is proportional to the energy transmitted to infinity in one cycle. By dividing by the mass of displaced fluid, both quantities can be made non-dimensional and are known as the virtual-mass and wave-making (or damping) coefficients. They measure respectively the components of force which are in and out of phase with the given acceleration of the heaving body.

However, the motion of rolling or pitching about an axis in the free surface has features similar to those of heaving motion. Suppose that the body rotates with angular velocity $\Omega\,e^{-i\sigma t}$ about the $+y$ axis. Then the normal derivative u in (1.3) is given by

$$\left(\frac{\partial\phi}{\partial n}\right)_C = u = \Omega\left(x\frac{\partial z}{\partial n} - z\frac{\partial x}{\partial n}\right)_C \tag{1.6}$$

and the moment of the pressure forces acting on the cylinder against the angular acceleration $\mathrm{Re}\,[-i\sigma\Omega\,e^{-\sigma t}]$ is

$$\mathrm{Re}\left[i\sigma\rho\,e^{-i\sigma t}\int_C \phi\left(x\frac{\partial z}{\partial n} - z\frac{\partial x}{\partial n}\right)dS\right].$$

Hence the added moment I is given by

$$I = -\frac{\rho}{\Omega^2}\,\mathrm{Re}\int_C \phi\,\frac{\partial\phi}{\partial n}\,dS. \tag{1.7}$$

This can be compared with the moment of inertia about the y axis of the displaced fluid to obtain an added moment-coefficient.

In this paper only short waves are considered, in which case K is large and the surface wave disturbance is essentially confined to a layer of thickness $O(K^{-1})$ below the free surface. But since K^{-1} is a length, K large has no absolute meaning without reference to a length scale of the problem. Hence K^{-1} is to be small compared with the girth of the body and the values of the vertical radii of curvature of C near F. It is on this basis that the length scale l is chosen for the definition of the dimensionless, large parameter N, given by

$$N = Kl. \tag{1.8}$$

Since the disturbance is essentially confined near the free surface, it is expected and can be shown that there is a negligible difference between corresponding situations with infinite depth and finite, constant depth. The former case will be usual here since it is simpler to handle, and the modifications required for finite depth will be discussed in § 5.

Considering first two-dimensional problems, the simplest of this type, namely heaving motion of an infinitely long, half-immersed, horizontal circular cylinder, was studied rigorously by Ursell (1953). Later (1954) he suggested plausible arguments for obtaining the same results more simply, using them to show that the wave-making effect of a rectangular cylinder is exponentially small and emphasizing the cylinder's barrier effect, which implied that the results of some previous authors must be in error. The plausible arguments showed that the leading terms of the virtual-mass and wave-making coefficients depend only on the limit potential ϕ_0, which satisfies (1.1) and (1.3) and vanishes on F and at infinity. This was subsequently established rigorously by Rhodes-Robinson (1970, 1972) for heaving motion in water of finite or infinite depth of a cylinder of arbitrary cross-section having vertical tangents and finite curvature at the free surface.

Rigorous results for an oscillating finite dock were obtained by Holford (1964) and further terms were found, by means of a different approach, by Leppington (1970).

Holford (1965) summarized how previous work on a heaving cylinder with an arbitrary angle of intersection α of C and F had failed to give correct limiting values when α tends to zero or $\frac{1}{2}\pi$. By means of suitable assumptions involving the use of known sloping-beach potentials, Holford obtained a formula for the wave amplitude at infinity which agrees with the dock result and becomes singular as $\alpha \to \frac{1}{2}\pi$. Such a singularity is reasonable, since when $\alpha = \frac{1}{2}\pi$, the heaving motion does not, to first order, have any wave-making effect, a fact which is exploited in the analysis for this situation by using the limit potential ϕ_0 defined above.

Apart from having one less variable, two-dimensional problems have the added advantages compared with their three-dimensional counterparts of the availability of complex-variable techniques and the fact that the body acts essentially as a barrier between semi-infinite regions of fluid. Thus the rigorous methods used for the two-dimensional dock do not work in three dimensions, where there is also no equivalent of the sloping-beach solution. However the plausible method of matched asymptotic expansions used for two-dimensional problems by Leppington (1972, 1973*a*) has been successfully applied by the same author (1973*b*) to a heaving or rolling circular dock and a heaving half-immersed sphere.

Other progress in three dimensions has been made only for cases where C meets F at right angles, which situation is assumed hereafter. Linearization of the boundary condition (1.3) implies that only the mean position of C is relevant. The rigorous methods of Ursell (1953) were adapted for a heaving half-immersed sphere by Davis (1971), who later (1975*a*) explained why extension of such methods to other bodies is impossible. However they can be modified to consider a heaving sphere in a hemispherical lake (Davis 1975*b*). With assumptions like those of Ursell (1954), Rhodes-Robinson (1971) obtained results for a heaving axisymmetric body and these have been used to consider the heaving torus (Davis 1975*c*).

Hermans (1973) has shown how ray methods can be used to consider the propagation of the waves generated by an arbitrarily shaped smooth body. Here, of course, the large parameter depends on frequency, and the application of ray methods must be distinguished from that discussed by Shen (1975), with an extensive bibliography, in which the large parameter is the ratio of horizontal and vertical length scales.

Considering now the present paper, §2 contains a summary, with two illustrative examples, of the two-dimensional problem. The general results in §3, obtained principally by suitable applications of Green's theorem, are asymptotic forms of (i) the upward force and the moment about the y axis of the pressure forces on the body in terms of limit potentials and (ii) the amplitudes of the cylindrical waves radiated to infinity in terms of a two-dimensional solution of the Helmholtz equation.

These results clearly remain applicable, in the absence of resonance, when there is present more than one body of the type considered. That they remain valid when the depth is finite is demonstrated in §5. In §4, their application to a heaving, rolling or pitching horizontal ellipsoid is considered and some numerical calculations tabulated. The results obtained are compared with those predicted by strip theory, used by naval engineers.

2. The two-dimensional problem

Here the immersed body is an infinitely long horizontal cylinder of uniform cross-section and the velocity potential ϕ is independent of a horizontal coordinate, say y. Then (1.1) can be written as

$$\partial^2\phi/\partial x^2 + \partial^2\phi/\partial z^2 = 0 \tag{2.1}$$

and the radiation condition takes the form

$$\phi \sim A^{\pm} e^{\pm iKx} e^{-Kz} \quad \text{as} \quad x \to \pm\infty. \tag{2.2}$$

Suppose that the boundary C, now identified by its cross-sectional curve, meets F at $(\pm a, 0)$ and that at these points the curvature of C is not large and the values of u in (1.3) are u_0^+ and u_0^- on the right and left respectively. Then, choosing the length scale l to be a, (1.8) becomes $N = Ka$.

For the commonly considered heaving motion, $u_0^+ = u_0^- = 0$ and hence it is of great help to introduce the limit potential ϕ_0, which satisfies (2.1), (1.3) and the limiting forms of (1.2) and the radiation condition as $K \to \infty$, namely

$$\phi_0 = 0 \quad \text{at} \quad z = 0 \quad \text{and at infinity.} \tag{2.3}$$

If one then writes

$$\phi = \phi_0 - \frac{1}{K}\frac{\partial \phi_0}{\partial z} + \frac{1}{N}\phi_1 \tag{2.4}$$

it is readily seen that ϕ_1 satisfies (2.1), radiation conditions like (2.2) and the boundary conditions (1.2) and

$$\frac{\partial \phi_1}{\partial n} = a\frac{\partial}{\partial n}\left(\frac{\partial \phi_0}{\partial z}\right) = v \quad \text{say} \quad \text{on } C. \tag{2.5}$$

Thus ϕ_1 satisfies a similar boundary-value problem to that for ϕ, except that now the values $v_0^{\pm}$ of v at $(\pm a, 0)$ are in general non-zero. Since ϕ_0 is wave free, all the wave terms in ϕ appear in ϕ_1 and it is seen that the lack of a wave-making effect, to first order, due to $u_0^{\pm}$ being zero, has led to the N^{-1} factor multiplying all wave terms. Rhodes-Robinson (1970) has proved that, in a heaving motion, the added mass M per unit length and the amplitudes $A^{\pm}$ are asymptotically given by

$$M \sim -\frac{\rho}{U^2}\left[\int_C \phi_0 \frac{\partial \phi_0}{\partial n} dS + \frac{1}{K}\int_F \left(\frac{\partial \phi_0}{\partial z}\right)^2 dx\right], \tag{2.6}$$

$$A^{\pm} \sim -(2ia/N^2)\, v_0^{\pm}\, e^{-iN}. \tag{2.7}$$

All the terms here depend only on the limit potential ϕ_0, which, being wave free, is much easier to find than ϕ. Similar results hold for any motion such that $u_0^+ = u_0^- = 0$, e.g. rolling. If either or both of these velocities is non-zero, then on the corresponding side(s) of the cylinder, ϕ is already like the ϕ_1 introduced above and the amplitudes are given by

$$A^+ \sim -\frac{2ia}{N} u_0^+ e^{-iN} \quad (u_0^+ \neq 0), \quad A^- \sim -\frac{2ia}{N} u_0^- e^{-iN} \quad (u_0^- \neq 0). \tag{2.8}$$

Note that these amplitudes can be obtained by writing $\phi = e^{-Kz}\psi(x)$ and solving $\psi''(x) + K^2\psi = 0$ in $|x| > a$ subject to the conditions

$$\psi \sim A^{\pm} e^{iK|x|} \quad \text{as} \quad |x| \to \infty, \quad \psi'(\pm a) = \pm 2u_0^{\pm}.$$

Similarly for ϕ_1 and formulae (2.7) when applicable.

As already implied, the orders of magnitude of $A^{\pm}$ are independent of $u_0^{\mp}$ respectively. Indeed the proofs of the formulae (2.7) and (2.8) can be adapted to the case of many parallel cylinders oscillating without resonance with the same frequency σ and all having boundaries which are vertical at $z = 0$. Then to leading order each far-field solution is unaware of the cylinders beyond the nearest one, in agreement with Ursell's (1961) result for the transmission of waves under obstacles.

There follow two illustrative examples.

(i) *The horizontally oscillating cylinder.* In this case, $u_0^{\pm} = \pm U$, and (2.8) is applicable, yielding $A^{\pm} \sim (\mp 2ia/N)\, U e^{-iN}$.

(ii) *The rolling elliptic cylinder with (comparable) semi-axes a and b.* In terms of elliptic co-ordinates (ξ, η), the limit potential ϕ_0 is here given by

$$\phi_0 = \tfrac{1}{4}\Omega(b^2 - a^2)\exp[-2(\xi - \xi_0)]\sin 2\eta,$$

where $\xi = \xi_0$ on the cylinder. Then, since (2.6) applies to (1.7) as well as (1.5), the added moment per unit length of the cylinder is given asymptotically by

$$I \sim \tfrac{1}{16}\pi\rho(b^2-a^2)^2\left[1-\frac{16}{\pi K(b+a)}\left(\frac{\tan^{-1}q-q+\frac{1}{3}q^3}{q^5}\right)\right], \qquad (2.9)$$

where $q = (b-a)^{\frac{1}{2}}(b+a)^{-\frac{1}{2}}$ and is real or pure imaginary according as $b \gtrless a$. By considering $v_0^{\pm}$, it follows on substitution in (2.7) and (2.2) that

$$\phi \sim \pm\frac{i\Omega a(b^2-a^2)(2b+a)}{N^2b^3}e^{iK(|x|-a)}e^{-Kz} \quad \text{as} \quad |x|\to\infty.$$

3. The three-dimensional problem: some general results

Equation (1.1) is now

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial z^2} = 0 \qquad (3.1)$$

and the radiation condition takes the form

$$\phi \sim \tfrac{1}{2}\sum_{n=0}^{\infty}\epsilon_n(A_n\cos n\theta+B_n\sin n\theta)H_n^{(1)}(KR)e^{-Kz} \quad \text{as} \quad (x^2+y^2)^{\frac{1}{2}} = R\to\infty, \qquad (3.2)$$

where ϵ_n is Neumann's symbol ($\epsilon_0 = 1, \epsilon_n = 2$ when $n \geqslant 1$). Let C_0 denote the intersection of C and F and $u_0(s)$ the values of the normal velocity u in (1.3) on this closed curve. Denote by $\|u_0\|$ the L^∞ norm of $u_0(s)$, i.e. the maximum value of $|u_0(s)|$, and assume first that $\|u_0\| > 0$.

An application of Green's theorem to ϕ and its complex conjugate $\bar{\phi}$ in the fluid region shows, in virtue of the Wronskian

$$\left[H_n^{(2)}\frac{dH_n^{(1)}}{dx}-H_n^{(1)}\frac{dH_n^{(2)}}{dx}\right]_{x=KR} = \frac{4i}{\pi KR}, \qquad (3.3)$$

that

$$\operatorname{Im}\left[-\pi\rho\int_C\phi\frac{\partial\bar{\phi}}{\partial n}dS\right] = \frac{\pi\rho}{2K}\sum_{n=0}^{\infty}\epsilon_n(|A_n|^2+|B_n|^2) \qquad (3.4)$$

$$= \text{energy transmitted in one cycle.}$$

Consider the function $\psi(x,y)$ satisfying

$$\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}+K^2\psi = 0 \quad \text{outside} \quad C_0 \qquad (3.5)$$

and the conditions

$$\partial\psi/\partial n = 2u_0(s) \quad \text{on} \quad C_0, \qquad (3.6)$$

$$\psi \sim \tfrac{1}{2}\sum_{n=0}^{\infty}\epsilon_n(a_n\cos n\theta+b_n\sin n\theta)H_n^{(1)}(KR) \quad \text{as} \quad R\to\infty. \qquad (3.7)$$

Note that this is analogous to the ψ defined in §2. Assuming that, as $N\to\infty$,

$$\phi|_C \sim o(l\|u_0\|) \quad \text{as} \quad z/l\to 0, \qquad (3.8)$$

it will now be shown that the amplitudes $\{A_n, B_n\}$ are asymptotically equal to $\{a_n, b_n\}$. (This confirms the conjecture made by Leppington (1973*b*) concerning the scattering of surface waves since that problem is easily reduced to the present one.)

Let $H(x,y,z) = H_0(x,y)\,e^{-Kz}$ be a general combination of incoming and outgoing cylindrical waves such that $\partial H/\partial n$ vanishes on C_0, i.e.

$$H_0(x,y) = \frac{1}{2}\sum_{n=0}^{\infty} \epsilon_n(c_n \cos n\theta + d_n \sin n\theta)\, H_n^{(2)}(KR) + \frac{1}{2}\sum_{n=0}^{\infty} \epsilon_n(C_n \cos n\theta + D_n \sin n\theta)\, H_n^{(1)}(KR), \tag{3.9}$$

$$\partial H_0/\partial n = 0 \quad \text{on} \quad C_0. \tag{3.10}$$

The coefficients $\{C_n, D_n\}$ are uniquely determined from $\{c_n, d_n\}$ and vice versa. By considering complex conjugates, the infinite matrix relating these coefficients is seen to be unitary.

Let Green's theorem be applied to H_0 and ψ outside C_0. The functions satisfy (3.5) and the conditions (3.6) and (3.10) and their behaviour at infinity is given by (3.7) and (3.9). The contributions from infinity are evaluated by means of (3.3), yielding

$$\int_{C_0} H_0 u_0\, ds = i \sum_{n=0}^{\infty} \epsilon_n(a_n c_n + b_n d_n). \tag{3.11}$$

According to ray theory, the function $H_0(x,y)$ can be continued within C_0 as far as caustics, none of which intersect C_0 and whose positions depend only on the geometry of C_0. Thus $H(x,y,z)$ exists on and outside a vertical cylinder Γ, independent of K, enclosing the z axis and with cross-section lying strictly inside C_0. The continuation actually required below is $\gg K^{-2}l^{-1}$, i.e. asymptotically small. Let C^* be the part of C lying outside Γ.

When Green's theorem is applied to H and ϕ in the fluid region outside Γ, then in virtue of (1.3) and the usual evaluation of the contribution from infinity, it follows that

$$K\int_{C^*}\left(Hu - \phi\frac{\partial H}{\partial n}\right) dS \sim i \sum_{n=0}^{\infty} \epsilon_n(A_n c_n + B_n d_n), \tag{3.12}$$

the error being exponentially small, namely the contribution from Γ.

Consider the integral on the left-hand side of (3.12). Since $H = H_0(x,y)\,e^{-Kz}$, only points on C up to a depth $O(K^{-1})$ need be considered. Since C is vertical at F, this leaves a thin strip bounded by C_0 and such that variations in KR in any vertical plane are $O(Kz^2/l)$, i.e. small. It is the rapid exponential decay combined with this slow variation of the corresponding oscillatory terms which is so vital in these problems. With H_0 given by (3.9) and satisfying (3.10), it can be shown that

$$H|_C \sim H_0|_{C_0}[1 + O(Kz^2/l)]\,e^{-Kz},$$
$$[\partial H/\partial n]_C \sim H_0|_{C_0}\,O(K^2z^2/l)\,e^{-Kz}.$$

Hence, in virtue of (3.8), the left-hand side of (3.12) is asymptotically equal to that of (3.11), for all sets of coefficients $\{c_n, d_n\}$. Thus the same must be true of the right-hand sides, implying that

$$a_n \sim A_n, \quad b_n \sim B_n \quad \text{for all } n.$$

So the amplitudes in (3.2) and hence the behaviour of ϕ as $R \to \infty$ can be asymptotically determined by solving the simpler, two-dimensional problem given by (3.5)–(3.7) to which ray theory can be applied. The surface elevation is then given by (1.4).

It is crucial to the above argument that all possible sets of coefficients are considered. As $\{c_n, d_n\}$ are varied, the order of magnitude of the leading term in

$$\int_{C_0} H_0 u_0 \, ds$$

is established; similarly for $\{C_n, D_n\}$ and the left-hand side of (3.12). The result is obtained by equating these leading terms, it then being immaterial that they may vanish for particular sets of coefficients. Rhodes-Robinson (1971), with only one coefficient to find in the axisymmetric case and with the normal velocity real, was able to construct an H which is non-oscillatory on C_0, namely $\psi - \overline{\psi}$, and thus ensure the non-vanishing of the leading terms above.

As a simple example, consider a horizontally vibrating sphere of radius a. Writing $x = r \sin\alpha \cos\theta$ and $z = r\cos\alpha$, one has $u_0(\theta) = U\cos\theta$ and then

$$\phi \sim \frac{2U}{K}\left[\frac{dH_1^{(1)}}{dx}(N)\right]^{-1} H_1^{(1)}(KR)\, e^{-Kz}\cos\theta \quad \text{as} \quad R \to \infty$$

$$\sim -\frac{2Ui}{K}\left(\frac{a}{R}\right)^{\frac{1}{2}} e^{iK(R-a)}\, e^{-Kz}\cos\theta \quad \text{as} \quad R \to \infty$$

after simplifying by means of the asymptotic form of the Hankel function. The energy radiated in one cycle, given by (3.4), is $2\rho\pi^2U^2a^3N^{-2}$, where $N = Ka$.

In the case when $\|u_0\| = 0$, i.e. $u_0(s) \equiv 0$, the limit potential ϕ_0 is used, as in §2, to introduce the subsidiary potential ϕ_1 and the wave terms are uniformly scaled by N^{-1}. The same argument as above can now be applied to ϕ_1. Thus if $v_0(s)$ denotes the values of v, given by (2.5), on C_0, then assuming that as $N \to \infty$

$$\phi_1|_C \sim o(l\|v_0\|) \quad \text{as} \quad z/l \to 0, \tag{3.13}$$

the coefficients $\{A_n, B_n\}$ in (3.2) are asymptotically given by $\{a_n, b_n\}$ in the solution of (3.5) and (3.7) and

$$\partial\psi/\partial n = N^{-1}v_0(s) \quad \text{on} \quad C. \tag{3.14}$$

So $\{A_n, B_n\}$ are reduced by $O(N^{-1})$ and the energy radiated in one cycle now normally has a factor N^{-4}, as indicated by the wave-making coefficient for a heaving sphere (Davis 1971).

Though the result here concerns the behaviour at infinity, it is reasonable to suppose that the solution at any given field point P not close to the body is asymptotically given by the corresponding solution of the two-dimensional Helmholtz equation, which is obtained by the ray-theory method (Keller, Lewis & Seckler 1956). Indeed Ursell (1966) found that the dominant contribution comes from point(s) on the body whose normal passes through P. Hermans (1973) used ray theory directly and his presentation is improved by the observation that his normal derivative is equivalent to the function $v = l\partial^2\phi_0/\partial n\,\partial z$ defined in (2.5) here.

Consider now the upward force on the body and the moment of this force about the y axis, for a general motion leading to a potential ϕ. These are conveniently expressed as

$$\frac{i\sigma\rho}{U}e^{-i\sigma t}\int_C \frac{\partial\phi^H}{\partial n}dS, \quad \frac{i\sigma\rho}{\Omega}e^{-i\sigma t}\int_C \phi\frac{\partial\phi^P}{\partial n}dS,$$

the potentials ϕ^H and ϕ^P, for pure heaving and rolling (or pitching), being appropriate here because they satisfy the boundary conditions

$$\frac{\partial\phi^H}{\partial n} = U\frac{\partial z}{\partial n}, \quad \frac{\partial\phi^P}{\partial n} = \Omega\left(x\frac{\partial z}{\partial n} - z\frac{\partial x}{\partial n}\right) \quad \text{on } C \tag{3.15}$$

and thus give rise to the correct integrals of pressure. Since these derivatives vanish on C_0, the limit potentials ϕ_0^H and ϕ_0^P can be used to introduce the potentials ϕ_1^H and ϕ_1^P as in (2.4). Corresponding to (2.5), one has

$$\frac{\partial\phi_1^H}{\partial n} = l\frac{\partial}{\partial n}\left(\frac{\partial\phi_0^H}{\partial z}\right), \quad \frac{\partial\phi_1^P}{\partial n} = l\frac{\partial}{\partial n}\left(\frac{\partial\phi_0^P}{\partial z}\right). \tag{3.16}$$

The limit potentials ϕ_{10}^H and ϕ_{10}^P of ϕ_1^H and ϕ_1^P respectively satisfy (2.3) and (3.16), and it will be assumed, as done by Rhodes-Robinson (1971), that

$$\phi_1^H \sim \phi_{01}^H, \quad \phi_1^P \sim \phi_{10}^P \quad \text{as} \quad Kz \to \infty. \tag{3.17}$$

By applying Green's theorem in the fluid region to (i) ϕ and ϕ^H and (ii) ϕ_0 and $\phi_{10}^H - l\,\partial\phi_0^H/\partial z$, it follows using the relevant boundary conditions that

$$\int_C \phi\frac{\partial\phi^H}{\partial n}dS = \int_C \phi^H\frac{\partial\phi}{\partial n}dS = \int_C \phi^H u\,dS,$$

$$\frac{1}{N}\int_C\left(\phi_{10}^H - l\frac{\partial\phi_0^H}{\partial z}\right)u\,dS = \frac{1}{K}\int_F \frac{\partial\phi_0^H}{\partial z}\frac{\partial\phi_0}{\partial z}dS.$$

Also the assumption (3.17) implies that

$$\int_C \phi_1^H u\,dS \sim \int_C \phi_{10}^H u\,dS$$

and hence

$$\int_C \phi\frac{\partial\phi^H}{\partial n}dS \sim \int_C \phi_0^H u\,dS + \frac{1}{K}\int_F \frac{\partial\phi_0^H}{\partial z}\frac{\partial\phi_0}{\partial z}dS,$$

where the leading term requires knowledge only of ϕ_0^H. Similarly

$$\int_C \phi\frac{\partial\phi^P}{\partial n}dS \sim \int_C \phi_0^P u\,dS + \frac{1}{K}\int_F \frac{\partial\phi_0^P}{\partial z}\frac{\partial\phi_0}{\partial z}dS.$$

The particular cases $\phi = \phi^H$ and $\phi = \phi^P$ yield, on substitution in (1.5) and (1.7),

$$\left.\begin{aligned} M &\sim -\frac{\rho}{U^2}\left[\int_C \phi_0^H\frac{\partial\phi_0^H}{\partial n}dS + \frac{1}{K}\int_F\left(\frac{\partial\phi_0^H}{\partial z}\right)^2 dS\right], \\ I &\sim -\frac{\rho}{\Omega^2}\left[\int_C \phi_0^P\frac{\partial\phi_0^P}{\partial n}dS + \frac{1}{K}\int_F\left(\frac{\partial\phi_0^P}{\partial z}\right)^2 dS\right]. \end{aligned}\right\} \tag{3.18}$$

For example, consider the motion generated when C is a sphere of radius a and the function u is given by

$$\frac{\partial\phi}{\partial n} = u = \sum_{m=0}^{1}\sum_{n=1}^{3} u_{mn} P_n^m(\cos\alpha)\cos m\theta,$$

where $z = r\cos\alpha$ and $x = r\sin\alpha\cos\theta$. The limit potentials are easily found to be

$$\phi_0^H = (-Ua^3/2r^2)\cos\alpha, \quad \phi_0^P \equiv 0,$$

$$\phi_0 = -\sum_{m=0}^{1}\sum_{n=1}^{3}\frac{u_{mn}}{n+1}\frac{a^{n+2}}{r^{n+1}} P_n^m(\cos\alpha)\cos m\theta,$$

and after some elementary calculation, the upward force on the sphere is

$$\sim -i\sigma\rho\pi a^3 e^{-i\sigma t}[\tfrac{1}{3}u_{01} + \tfrac{1}{8}u_{02} - (8N)^{-1}(u_{01} - \tfrac{1}{2}u_{03})],$$

whilst the moment of this force about the y axis is zero. Also, with $R = r\sin\alpha$ and $N = Ka$ and again using asymptotic forms to simplify the Hankel functions,

$$\phi \sim N^{-1}(aR)^{\frac{1}{2}} e^{-Kz} e^{iK(R-a)}[2u_{11}\cos\theta - u_{02}\cos 2\theta - 3u_{13}\cos 3\theta] \quad \text{as} \quad R\to\infty$$

and the energy transmitted in one cycle is

$$\tfrac{1}{2}\pi^2\rho a^3 N^{-2}[4|u_{11}|^2 + |u_{02}|^2 + 9|u_{13}|^2].$$

However if $u_{11} = u_{02} = u_{13} = 0$, the wave making is due to the subsidiary potential ϕ_1, in which case further calculation shows that

$$\phi \sim N^{-2}(aR)^{\frac{1}{2}} e^{-Kz} e^{iK(R-a)}[3u_{01} - \tfrac{15}{4}u_{03} + 8u_{12}\cos\theta] \quad \text{as} \quad R\to\infty$$

and the energy transmitted in one cycle is

$$\pi^2\rho a^3 N^{-4}[9|u_{01} - \tfrac{5}{4}u_{03}|^2 + 32|u_{12}|^2].$$

These are not the second terms in expansions beginning with the expressions above. In heaving motion, only u_{01} is non-zero whilst in the example considered earlier, the horizontally oscillating sphere, only u_{11} is present.

4. The heaving and rolling of a horizontal ellipsoid

This section considers an ellipsoid whose axis lies along the x axis and whose boundary C is given by

$$\frac{x^2}{a^2} + \frac{y^2+z^2}{b^2} = 1. \tag{4.1}$$

(The alignment of the horizontal axis is immaterial in heaving motion, whilst if there is symmetry about the y axis, then rolling produces no waves. Also not considered is the ellipsoid with vertical axis of symmetry because, with some modification for rolling, it can be treated by the methods of Rhodes-Robinson (1971). Rotations about the y axis ,as in (1.6), are described as rolling or pitching according as $a \lesssim b$. If the usual elliptic co-ordinates are used, as in example (ii) of §2, then separate calculations are required for the cases $a \lesssim b$, although the steps are identical. To avoid this duplication, define (λ, η, θ) by

$$x = \lambda\cos\eta, \quad (y,z) = (\lambda^2 + b^2 - a^2)^{\frac{1}{2}}\sin\eta(\sin\theta, \cos\theta). \tag{4.2}$$

Then the fluid region is $\lambda \geqslant a$, $0 \leqslant \eta \leqslant \pi$ and $|\theta| \leqslant \frac{1}{2}\pi$. The line elements $h_\lambda d\lambda$, $h_\eta d\eta$ and $h_\theta d\theta$ are given by

$$h_\lambda = \left[\frac{\lambda^2+(b^2-a^2)\cos^2\eta}{\lambda^2+b^2-a^2}\right]^{\frac{1}{2}}, \quad h_\eta = [\lambda^2+(b^2-a^2)\cos^2\eta]^{\frac{1}{2}}, \quad h_\theta = (\lambda^2+b^2-a^2)^{\frac{1}{2}}\sin\eta \tag{4.3}$$

and hence (1.1) becomes

$$\frac{\partial}{\partial\lambda}\left[(\lambda^2+b^2-a^2)\frac{\partial\phi}{\partial\lambda}\right]+\frac{1}{\sin\eta}\frac{\partial}{\partial\eta}\left(\sin\eta\frac{\partial\phi}{\partial\eta}\right)+\left(\frac{1}{\sin^2\eta}-\frac{b^2-a^2}{\lambda^2+b^2-a^2}\right)\frac{\partial^2\phi}{\partial\theta^2} = 0. \tag{4.4}$$

Particular solutions of (4.4) are z and xz and here the required solutions, which must vanish at infinity, will be of the form

$$\phi = F(\lambda)z \quad \text{or} \quad \phi = G(\lambda)xz.$$

F and G can be expressed in terms of Legendre functions, but it is simpler to solve directly, obtaining

$$F(\lambda) = -Ab^2a\int_\lambda^\infty \frac{d\nu}{(\nu^2+b^2-a^2)^2}, \quad G(\lambda) = Cb^2a(b^2-a^2)\int_\lambda^\infty \frac{d\nu}{\nu^2(\nu^2+b^2-a^2)^2}. \tag{4.5}$$

The potentials ϕ^H and ϕ^P of the heaving and rolling motions respectively are such that their limit potentials ϕ_0^H and ϕ_0^P vanish on F and satisfy (4.4) and the conditions (3.15), i.e.

$$\partial(\phi_0^H - Uz)/\partial\lambda = 0 \quad \text{at} \quad \lambda = a,$$

$$\frac{\partial\phi_0^P}{\partial\lambda} = \Omega\left(x\frac{\partial z}{\partial\lambda}-z\frac{\partial x}{\partial\lambda}\right) \quad \text{at} \quad \lambda = a.$$

The appropriate solutions are of the form

$$\phi_0^H = UF(\lambda)z, \quad \phi_0^P = \Omega G(\lambda)xz, \tag{4.6}$$

where F and G are given by (4.5), in which case the conditions at $\lambda = a$ become

$$F'(a) = [1-F(a)]a/b^2, \quad G'(a) = (ab^2)^{-1}[a^2-b^2-(a^2+b^2)G(a)].$$

On substituting (4.5), equations determining A and C are obtained, namely

$$A^{-1} = 1-ab^2\int_a^\infty \frac{d\nu}{(\nu^2+b^2-a^2)^2},$$

$$C^{-1} = 1-ab^2(a^2+b^2)\int_a^\infty \frac{d\nu}{\nu^2(\nu^2+b^2-a^2)^2}. \tag{4.7}$$

Now from (4.6)

$$\left.\frac{\partial\phi_0^H}{\partial z}\right|_{z=0} = UF(\lambda), \quad \left.\frac{\partial\phi_0^P}{\partial z}\right|_{z=0} = \Omega G(\lambda)x, \tag{4.8}$$

and on substitution in (3.18), with the surface elements given by (4.3) and F and G by (4.5), the free-surface integrals can be evaluated by repeated integration by parts. After some calculation, one finds that

$$\frac{M}{\frac{2}{3}\pi\rho ab^2} \sim A-1-\frac{1}{Kb}\left\{A^2\left[\tfrac{1}{2}+\left(\frac{b}{a+b}\right)^2\right]-\tfrac{3}{2}\right\},$$

i.e.
$$V \sim \beta_0 - \beta_1/Kb, \tag{4.9}$$
where V is the virtual-mass coefficient, and
$$\frac{I}{\rho\pi ab^2} \sim \frac{2}{15}\frac{(a^2-b^2)^2}{(a^2+b^2)}(C-1) + \frac{1}{4Kb}\left(\frac{a^2-b^2}{a^2+b^2}\right)^2[a^2+2b^2C-(a^2+2b^2)C^2] + \frac{C^2}{6Kb}(a-b)^2\left(\frac{a+3b}{a+b}\right), \tag{4.10}$$
i.e. since the moment of inertia of the immersed body about the y axis is $\frac{2}{15}\rho\pi ab^2(a^2+b^2)$, the added-moment coefficient $\sim \gamma_0$, where
$$\gamma_0 = \left(\frac{a^2-b^2}{a^2+b^2}\right)^2(C-1).$$
These dimensionless quantities are introduced for convenience of tabulation below. From (4.7), it may be shown that
$$\left.\begin{aligned} C^{-1} &= 1+\left(\frac{a^2+b^2}{a^2-b^2}\right)(3A^{-1}-2), \\ A^{-1} &= \begin{cases} \dfrac{b^2a}{4c^3}\log\dfrac{a+c}{a-c}+1-\dfrac{a^2}{2c^2} & (a > b,\quad c^2 = a^2-b^2), \\ \dfrac{1}{2}\left[1+\dfrac{b^2}{c^2}-\dfrac{ab^2}{c^3}\tan^{-1}\left(\dfrac{c}{a}\right)\right] & (a < b, c^2 = b^2-a^2). \end{cases}\end{aligned}\right\} \tag{4.11}$$
Computed values of the coefficients β_0, β_1 and γ_0 are given in table 1.

The reduction of the ratio b/a is limited by the fact that b is the depth of the body, a fact manifested by the appearance of Kb as the large parameter in the asymptotic expansions (4.9) and (4.10). A further length whose smallness must be borne in mind is the radius of curvature of C at the ends $(\pm a, 0, 0)$, namely b^2/a.

When $a/b \to 0$, $C^{-1} \to 0$ and γ_0 becomes infinite. However the added moment remains finite, having the limiting value $\frac{8}{45}\rho b^5$, because $Ca/b \to 4/3\pi$. This corresponds to the stabilizing effect of a keel on a boat.

The waves radiated to infinity can, according to §3, be determined by finding solutions ψ^H and ψ^P of (3.5) on F which, as in (3.7), have only outgoing waves at infinity and satisfy the appropriate forms of (3.14), which with the aid of (4.8) are
$$\left.\frac{\partial\psi^H}{\partial\lambda}\right|_{\lambda=a} = \frac{2UAa}{Kb^2}, \quad \left.\frac{\partial\psi^P}{\partial\lambda}\right|_{\lambda=a} = \frac{2\Omega}{Kb^2}\left(\frac{a^2-b^2}{a^2+b^2}\right)(b^2+a^2C)\cos\eta,. \tag{4.12}$$
It is convenient now to revert to the usual elliptic co-ordinates. For $a > b$, write
$$c^2 = a^2-b^2, \quad \lambda = c\cosh\xi, \quad \eta = \vartheta, \quad a = c\cosh\xi_0, \quad b = c\sinh\xi_0.$$
For $a < b$, write
$$c^2 = b^2-a^2, \quad \lambda = c\sinh\xi, \quad \eta = \tfrac{1}{2}\pi-\vartheta, \quad a = c\sinh\xi_0, \quad b = c\cosh\xi_0.$$
Then (3.5) becomes, in either case,
$$\frac{\partial^2\psi}{\partial\xi^2}+\frac{\partial^2\psi}{\partial\vartheta^2}+K^2c^2(\cosh^2\xi-\cos^2\vartheta)\psi = 0.$$

$a > b$: heaving and pitching

b^2/a^2	β_0	β_1	γ_0	C^{-1}
1	0·5000	0·1875	0	0·6000
0·9	0·5158	0·1935	0·0018	0·6028
0·8	0·5336	0·2003	0·0081	0·6052
0·7	0·5539	0·2083	0·0201	0·6072
0·6	0·5772	0·2176	0·0402	0·6086
0·5	0·6046	0·2290	0·0713	0·6090
0·4	0·6376	0·2434	0·1185	0·6079
0·3	0·6790	0·2625	0·1900	0·6041
0·2	0·7338	0·2902	0·3020	0·5954
0·1	0·8151	0·3375	0·4935	0·5756

(Exact values for the sphere included for comparison.)

$a < b$: heaving and rolling

a^2/b^2	β_0	β_1	γ_0	C^{-1}
0·9	0·4843	0·1816	0·0019	0·5967
0·8	0·4668	0·1752	0·0085	0·5924
0·7	0·4472	0·1681	0·0219	0·5867
0·6	0·4250	0·1602	0·0454	0·5791
0·5	0·3994	0·1511	0·0843	0·5686
0·4	0·3690	0·1405	0·1481	0·5536
0·3	0·3319	0·1275	0·2559	0·5312
0·2	0·2837	0·1107	0·4545	0·4944
0·1	0·2132	0·0856	0·9162	0·4222
0·04	0·1425	0·0593	1·7930	0·3221
0·01	0·0748	0·0323	4·0223	0·1928

TABLE 1

The solutions in separated variables are in terms of Mathieu functions. $\mathrm{Se}_n(Kc, \cos\vartheta)$ and $\mathrm{So}_n(Kc, \cos\vartheta)$ are respectively even and odd in ϑ with period π or 2π according as n is even or odd. The corresponding radial solutions for outgoing waves are $\mathrm{He}_n(Kc, \cosh\xi)$ and $\mathrm{Ho}_n(Kc, \cosh\xi)$. For a detailed discussion, including the notation, equations and properties quoted below, see Morse & Feshbach (1953, pp. 1407–1411, 1422, 1568–1572).

Now, since

$$\sum_{m=0}^{\infty} \frac{B_0^e(Kc, 2m)}{M_{2m}^e(Kc)} \mathrm{Se}_{2m}(Kc, \cos\vartheta) = \frac{1}{2\pi}$$

the solution for the heaving motion is

$$\psi^H = \frac{4\pi U A a}{Kb} \sum_{m=0}^{\infty} \frac{B_0^e(Kc, 2m)}{M_{2m}^e(Kc)} \mathrm{Se}_{2m}(Kc, \cos\vartheta) \frac{\mathrm{He}_{2m}(Kc, \cosh\xi)}{[d\mathrm{He}_{2m}(Kc, \cosh\xi)/d\xi]_{\xi=\xi_0}}.$$

When Kc is large, Se_{2m} decays exponentially away from $\vartheta = \pm\frac{1}{2}\pi$, i.e.the waves are propagated mainly at right angles to the major axis. At large values of $Kc\cosh\xi$, He_{2m} behaves like a Hankel function:

$$\mathrm{He}_{2m}(Kc, \cosh\xi) \sim (Kc\cosh\xi)^{-\frac{1}{2}} \exp\{i[Kc\cosh\xi - \tfrac{1}{2}\pi(2m+\tfrac{1}{2})]\},$$

i.e. the waves, as expected, ultimately become cylindrical. Further,

$$\mathrm{Se}_{2m}(Kc, \cos\eta)\,\mathrm{He}_{2m}(Kc, \cosh\xi) = (\tfrac{1}{2}\pi)^{\frac{1}{2}} \sum_{n=0}^{\infty} B^{e}_{2n}(Kc, 2m)\,\mathrm{H}^{(1)}_{2n}(KR)\cos 2n\theta$$

(θ being the polar angle of § 3), so that, in the notation of (3.7),

$$a_{2n-1} = b_n = 0 \quad (n \geqslant 1),$$

$$|\tfrac{1}{2}\epsilon_n a_{2n}| \sim \frac{2\pi UAa^2}{Kb^2}\left(\frac{2\pi}{Ka}\right)^{\frac{1}{2}}\left|\sum_{m=0}^{\infty}\frac{(-1)^m B^{e}_{0}(Kc, 2m)\,B^{e}_{2n}(Kc, 2m)}{M^{e}_{2m}(Kc)}\right|.$$

When $c \to 0$, $B^{e}_{2n}(Kc, 2m) \sim \delta_{mn}$ (the Kronecker delta) and $M^{e}_{2m}(Kc) \sim 2\pi/\epsilon_m$, and since $A \to \frac{2}{3}$, the known result for the heaving sphere is recovered. Notice that interchanging a and b does not affect the Mathieu functions and their constants but only the multiplicative factors of ψ^H.

Similarly, for the pitching and rolling motions, the expansions

$$\pi \sum_{m=0}^{\infty} \frac{B^{e}_{1}(Kc, 2m+1)}{M^{e}_{2m+1}(Kc)}\,\mathrm{Se}_{2m+1}(Kc, \cos\vartheta) = \cos\vartheta,$$

$$\pi \sum_{m=0}^{\infty} \frac{B^{o}_{1}(Kc, 2m+1)}{M^{o}_{2m+1}(Kc)}\,\mathrm{So}_{2m+1}(Kc, \cos\vartheta) = \sin\vartheta$$

may be respectively used. In either case, no waves propagate along the y axis and when Kc is large, the disturbance decays exponentially away from the x axis.

The ray-theory solution of the heaving problem shows that, at large distances R in the direction making an angle β with the x axis,

$$\psi^H \sim C(\beta)\,R^{-\frac{1}{2}}\exp(iKR),$$

where

$$C(\beta) = \frac{2UA}{i(Kb)^2}\left[\frac{1}{a^2}\sin^2\beta + \frac{1}{b^2}\cos^2\beta\right]^{-\frac{3}{4}}\exp\left[-iK(a^2\cos^2\beta + b^2\sin^2\beta)^{\frac{1}{2}}\right].$$

Finally it is appropriate to compare these results with those of the strip theory applied to slender bodies (Ogilvie & Tuck 1969). For $a \gg b$, (4.9) and (4.11) imply that

$$M \sim \tfrac{2}{3}\rho\pi b^2 a\left(1 - \frac{1}{2Kb}\right)\left[1 + O\left(\frac{b^2}{a^2}\log\frac{a}{b}\right)\right].$$

But using the circular-cylinder result of Ursell (1953), the strip theory predicts a virtual mass of

$$\int_0^{\frac{1}{2}\pi} \tfrac{1}{2}\rho\pi b^2 \sin^2\eta\left[1 - \frac{4}{3\pi Kb\sin\eta}\right] a\sin\eta\,d\eta = \tfrac{2}{3}\rho\pi ab^2\left(1 - \frac{1}{2Kb}\right)$$

in very good agreement, despite the restriction that Kb be large. The requirement $K \gg a/b^2$, the curvature at the 'sharp' ends of the body, appears to be unnecessary because these points make no contribution to the transmission of waves along the minor axis. Strip theory is inapplicable to pitching motion, which is far from being two-dimensional.

For $b \gg a$, one finds that

$$V \sim \frac{\pi}{4}\frac{a}{b}\left[1-\frac{3}{Kb}\left(1-\frac{8}{3\pi}\right)\right]\left[1+O\left(\frac{a}{b}\right)\right]$$

whilst, as stated earlier, the added moment $\sim \frac{8}{45}\rho b^5[1+O(a/b)]$. Notice that the errors are more significant than in the $a \gg b$ case and that the leading term in V is zero. Both these factors contribute to the poorer agreement with strip theory in this case. The cross-section $y = b\sin\eta$ of the ellipsoid is an ellipse with axes $a\cos\eta$ and $b\sin\eta$. Rhodes-Robinson (1970*a*) showed that the virtual-mass coefficient for an ellipse with axes a and b is

$$\sim\frac{a}{b}-\frac{8a}{\pi Kb(a+b)}\left[\frac{q-\tan^{-1}q}{q^3}\right], \quad \text{where} \quad q = \left(\frac{b-a}{b+a}\right)^{\frac{1}{2}},$$

whilst the corresponding result for the added moment is given by (2.9). Hence strip theory suggests that

$$V \sim \frac{a}{b}\left\{1-\frac{3}{K(a+b)}\left(\frac{q-\tan^{-1}q}{q^3}\right)\right\},$$

$$I \sim \tfrac{1}{15}\rho\pi bc^4 \sim \tfrac{1}{15}\rho\pi b^5.$$

Agreement for I is reasonable and for V the zero leading term is correctly predicted, but errors of some 25 % are present in the first non-zero terms of V.

These methods can also be usefully applied to a vibrating finite vertical strip. There is some simplification due to the boundary condition on the radial functions He_n and Ho_n being applied at $\xi = 0$. For a similar example, see Morse & Feshbach (1953, pp. 1423–1425).

5. Modifications for finite depth

In this case the additional condition

$$\partial\phi/\partial z = 0 \quad \text{at} \quad z = h \tag{5.1}$$

must be applied. This is satisfied by the limit potential ϕ_0 but not by $\partial\phi_0/\partial z$. Hence, following the method of Rhodes-Robinson (1971), the introduction of the subsidiary potential ϕ_0 in (2.4) must be modified as follows:

$$\phi = \phi_0-\frac{1}{K}\frac{\partial\phi_0}{\partial z}+\frac{1}{N}\phi_1+\frac{1}{K}\sum_{n=0}^{\infty}\epsilon_n\int_0^{\infty}\frac{k^2[a_n(k)\cos n\theta+b_n(k)\sin n\theta]}{k\sinh kh-K\cosh kh} \times (k\cosh ky-K\sinh ky)\,J_n(kR)\,dk, \tag{5.2}$$

where the integral has a contour indented below the pole $k = k_0$. The functions $a_n(k)$ and $b_n(k)$ are determined by (5.1), i.e.

$$\sum_{n=0}^{\infty}\epsilon_n\int_0^{\infty}k^3[a_n(k)\cos n\theta+b_n(k)\sin n\theta]\,J_n(kR)\,dK = \frac{\partial^2\phi}{\partial z^2}(R,\theta,h).$$

By means of (1.1), Fourier coefficients, inversion of Hankel transforms, integration by parts and Bessel's equation, it follows that

$$(a_n(k), b_n(k)) = \frac{1}{2\pi}\int_0^{\infty}\int_{-\pi}^{\pi}\phi_0(R,\theta,h)\,J_n(kR)\,(\cos n\theta, \sin n\theta)\,R\,dR\,d\theta.$$

Also, as $R \to \infty$,

$$\phi - \frac{1}{N}\phi_1 \sim \frac{2\pi i k_0^3 \cosh k_0(h-y)}{K(2k_0 h + \sinh 2k_0 h)} \sum_{n=0}^{\infty} \epsilon_n \left[a_n(k_0)\cos n\theta + b_n(k_0)\sin n\theta\right] H_n^{(1)}(k_0 R),$$

i.e. the correction term has exponentially small outgoing waves and so, asymptotically, the wave terms of ϕ are contained in ϕ_1.

Considering the leading terms, as $N \to \infty$,

$$v = \left(\frac{\partial \phi_1}{\partial n}\right)_C \sim l\left[\frac{\partial}{\partial n}\left\{\frac{\partial \phi_0}{\partial z} - \sum_{n=0}^{\infty} \epsilon_n \int_0^\infty \frac{k^2[a_n(k)\cos n\theta + b_n(k)\sin n\theta]}{\cosh kh} \times \sinh ky\, J_n(kR)\, dk\right\}\right]_C,$$

and in addition to the assumption (3.13), it is also assumed that

$$v \sim l\left[\frac{\partial}{\partial n}\left(\frac{\partial \phi_0}{\partial z}\right)\right]_C$$

as $N \to \infty$ and $z/l \to 0$. This form has the same appearance as the exact value of v in the infinite-depth case. Other corrections in § 3 are exponentially small and thus the results in terms of the limit potentials remain valid when the depth is finite.

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